

Physics 128 Discussion (Week 8)

Problem 1 (Graviton Decoupling)

Primordial gravitons are thought to decouple at $T \approx M_{\text{pl}} \approx 10^{19}$ GeV.

(a) Estimate the temperature today of the relic thermal graviton background assuming that only Standard Model particles are in thermal equilibrium with gravitons when they decouple. (The particles in the Standard Model add up to give $g_{*s} = 106.75$. The current temperature of the CMB is $T_0 = 0.24$ meV, and the current number of effective entropy degrees of freedom is $g_{*s,0} = 3.94$.)

After decoupling, gravitons “just redshift” as

$$a_{\text{decouple}} T_{\text{decouple}}^{\text{graviton}} = a_0 T_0^{\text{graviton}}. \quad (1)$$

After decoupling, standard model particles’ temperatures change due to the entropy conservation

$$g_{*s,\text{decouple}} (T_{\text{decouple}}^{\text{SM}})^3 a_{\text{decouple}}^3 = g_{*s,0} (T_0^{\text{SM}})^3 a_0^3. \quad (2)$$

At decoupling, temperature of gravitons is equal to temperature of standard model particles

$$T_{\text{decouple}}^{\text{SM}} = T_{\text{decouple}}^{\text{graviton}} \equiv T_D. \quad (3)$$

Thus we get

$$\begin{aligned} a_{\text{decouple}} T_D &= a_0 T_0^{\text{graviton}}, \\ g_{*s,\text{decouple}} T_D^3 a_{\text{decouple}}^3 &= g_{*s,0} (T_0^{\text{SM}})^3 a_0^3. \end{aligned} \quad (4)$$

When we combine these two, we get

$$\begin{aligned} T_0^{\text{graviton}} &= \left(\frac{g_{*s,0}}{g_{*s,\text{decouple}}} \right)^{1/3} T_0^{\text{SM}} \\ &= \left(\frac{3.94}{106.75} \right)^{1/3} (0.24 \text{ meV}) \\ &\simeq 0.08 \text{ meV}. \end{aligned} \quad (5)$$

(b) If we observed this relic graviton background, would we expect the temperature to be larger or smaller than the part (a) estimate?

If there are any extra degrees of freedom that are in thermal equilibrium with gravitons when they decouple, then $g_{*s,\text{decouple}} > 106.75$. This could occur in any theory in which there are extra heavy particles, such as grand unified theories (GUT). Then

$$T_0^{\text{graviton}} < 0.08 \text{ meV}. \quad (6)$$

We would expect the relic graviton background to be at a lower temperature than calculated in part (a).

(c) Estimate the number density of gravitons today under the assumptions of part (a).

From statistical mechanics, the number density of relativistic gravitons is given by Bose statistics

$$\begin{aligned} n &= g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{e^{E/T} - 1} \\ &= 2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{e^{E/T} - 1}. \quad (\text{Gravitons have two polarization states}) \\ &= \frac{1}{\pi^2} \int_0^\infty dp \frac{p^2}{e^{p/T} - 1} \quad (\because E = p) \\ &= \frac{T^3}{\pi^2} \int_0^\infty du \frac{u^2}{e^u - 1} \quad \left(u \equiv \frac{p}{T}\right) \\ &= \frac{2\zeta(3) T^3}{\pi^2} \\ &\simeq 0.244 T^3. \end{aligned} \quad (7)$$

Thus, we get

$$\begin{aligned} n_0^{\text{graviton}} &\simeq 0.244 T_0^{\text{graviton}} \\ &\simeq 0.244 (0.08 \times 10^{-3} \text{ eV})^3 \\ &\rightarrow 16.3 \text{ cm}^{-3}. \end{aligned} \tag{8}$$

(d) If we observed this relic graviton background, would we expect the number density of gravitons to be larger or smaller than the part (c) estimate?

Since

$$n \propto T^3, \tag{9}$$

and our part (a) estimate of T_0^{graviton} is an overestimate due to the GUT scale physics, our part (c) estimate of the number density is also an overestimate. We would expect the number density of gravitons to be smaller than the part (c) estimate.

Problem 2 (Self Interacting Dark Matter)

(a) Estimate the characteristic speed of dark matter particles bound in the dark halos of several structures in the universe using the Virial theorem

$$mv^2 \simeq \frac{GM_{\text{halo}}m}{R_{\text{halo}}}, \quad (10)$$

where m is DM particle mass, v is characteristic speed, and $M_{\text{halo}}, R_{\text{halo}}$ are the mass and radius of the structure that contains DM, respectively. Consider the case for

- A dwarf galaxy: $M \simeq 2 \times 10^6 M_{\odot}$, $R \simeq 1$ kpc

The viral theorem reads

$$\begin{aligned} mv^2 &\simeq \frac{GM_{\text{halo}}m}{R_{\text{halo}}} \\ \Rightarrow v &\simeq \sqrt{\frac{GM_{\text{halo}}}{R_{\text{halo}}}}. \end{aligned} \quad (11)$$

Then

$$v_{\text{dwarf}} \simeq 3.0 \times 10^3 \text{ m/s} \quad (12)$$

- The Milky way galaxy: $M \simeq 10^{12} M_{\odot}$, $R \simeq 100$ kpc

$$v_{\text{MW}} \simeq 2.1 \times 10^5 \text{ m/s} \quad (13)$$

- Galaxy cluster: $M \simeq 3 \times 10^{14} M_{\odot}$, $R \simeq 2$ Mpc

$$v_{\text{cluster}} \simeq 8.1 \times 10^5 \text{ m/s} \quad (14)$$

Self interacting dark matter (SIDM) must have a cross section very close to its upper limit in a particular type of structure to be substantially different from the usual collisionless cold dark matter. The ratio

$$\left(\frac{(\sigma_{\max}/m)}{6 \times 10^{-25} \text{ cm}^2 \text{ GeV}^{-1}} \right) \quad (15)$$

is between a few to 100 to have an effective core creation in dwarf galaxies, but is $\simeq 1$ in galaxy clusters. A constant cross section $\sigma_{\text{self}} = \sigma_{\max}$ which would be effective at dwarf galaxy scales would then be forbidden by the limits coming from galaxy clusters. But this may not be a problem if the cross section depends on the characteristic SIDM speed.

(b) Assuming the power-law cross section $\sigma_{\text{self}}(v) \sim v^{-n}$, which dependence could give $\sigma_{\text{self}}^{\text{dwarf}} \simeq 100 \times \sigma_{\text{self}}^{\text{cluster}}$? Assume that the cross section scales as a power-law

$$\sigma_{\text{self}}(v) \sim v^{-n}, \quad (16)$$

where n is the power-law index we need to determine. Then

$$\begin{aligned} \frac{\sigma_{\text{self}}(v_{\text{dwarf}})}{\sigma_{\text{self}}(v_{\text{cluster}})} &= \left(\frac{v_{\text{cluster}}}{v_{\text{dwarf}}} \right)^n \\ &\simeq \left(\frac{8.1 \times 10^5 \text{ m/s}}{3.0 \times 10^3 \text{ m/s}} \right)^n \\ &\simeq 10^{2n}. \end{aligned} \quad (17)$$

This implies that $n = 1$ to satisfy $\sigma_{\text{self}}^{\text{dwarf}} \simeq 100 \times \sigma_{\text{self}}^{\text{cluster}}$.

$$\therefore \sigma_{\text{self}}(v) \sim \frac{1}{v}. \quad (18)$$

(c) Suppose that the SIDM scattering is mediated by a very light gauge boson¹, then the cross section becomes similar to the Rutherford scattering, which is inversely proportional to the square of the kinetic energy

$$\sigma \sim v^{-4}. \quad (19)$$

Prove that if $\sigma_{\text{self}} \simeq \sigma_{\text{max}}$ in dwarfs, then σ_{self} would be $\ll \sigma_{\text{max}}$ in clusters (so SIDM becomes just CDM at large scales).

For the Rutherford-like scattering interaction, the cross section is given by

$$\sigma_{\text{self}}(v) \sim v^{-4} \quad (n = 4). \quad (20)$$

Thus the ratio is given by

$$\begin{aligned} \frac{\sigma_{\text{self}}(v_{\text{dwarf}})}{\sigma_{\text{self}}(v_{\text{cluster}})} &= \left(\frac{v_{\text{cluster}}}{v_{\text{dwarf}}} \right)^4 \\ &\simeq \left(\frac{8.1 \times 10^5 \text{ m/s}}{3.0 \times 10^3 \text{ m/s}} \right)^4 \\ &\simeq 10^8. \end{aligned} \quad (21)$$

From this, we get

$$\begin{aligned} \sigma_{\text{self}}(v_{\text{cluster}}) &\simeq 10^{-8} \sigma_{\text{self}}(v_{\text{dwarf}}) \\ &\ll \sigma_{\text{max}}. \end{aligned} \quad (22)$$

This shows that σ_{self} in cluster becomes many orders of magnitude smaller than its maximum value. For a cross section scaling as $\sigma_{\text{self}} \sim v^{-4}$, if σ_{self} is near σ_{max} in dwarf galaxies, it becomes much smaller in galaxy clusters. Therefore, SIDM effectively transitions to collisionless CDM at large scale.

¹In SIDM models, one has a “dark photon” (or “hidden photon”) instead of the usual photon but still if the mediator mass $m_\phi \ll m_{\text{DM}}$, then $\sigma_{\text{self}} \sim v^{-4}$.